Computers and Mathematics with Applications 64 (2012) 2603–2615

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A numerical study based on an implicit fully discrete local discontinuous Galerkin method for the time-fractional coupled Schrödinger system[☆]

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ARTICLE INFO

Article history:
Received 6 April 2012
Received in revised form 21 June 2012
Accepted 4 July 2012

Keywords:
Time-fractional coupled Schrödinger system
Local discontinuous Galerkin method
Stability
Error estimates

ABSTRACT

In this paper we develop and analyze an implicit fully discrete local discontinuous Galerkin (LDG) finite element method for solving the time-fractional coupled Schrödinger system. The method is based on a finite difference scheme in time and local discontinuous Galerkin methods in space. Through analysis we show that our scheme is unconditionally stable, and the L^2 error estimate has the convergence rate $O(h^{k+1}+(\Delta t)^2+(\Delta t)^{\frac{\alpha}{2}}h^{k+\frac{1}{2}})$ for the linear case. Extensive numerical results are provided to demonstrate the efficiency and accuracy of the scheme.

1. Introduction

In recent years the numerical modeling and simulation for fractional calculus have been stimulated due to their numerous applications in physics and engineering. Some applied scientists and engineers realized that many mathematical models are formulated in terms of fractional derivatives, which provided an excellent instrument for the description of memory and hereditary properties of various materials and processes. Scholars have been interested in researching the problems involving the fractional order partial differential equations [1–11].

Nonlinear coupled partial differential systems are very important in various fields, especially in fluid mechanics, solid state physics and plasma waves. The coupled Schrödinger system is used to model two interacting nonlinear packets in a dispersive and conservative system. Some methods [12–14] have been used to handle the integer-order systems; however, to the best of our knowledge, the study of the fractional coupled system has not been widespread. In this paper, we consider the following time-fractional coupled Schrödinger system

$$\begin{aligned} &iD_t^\alpha u(x,t)+iu_x+u_{xx}+u+v+\lambda_1 f(|u|^2,|v|^2)u=0,\\ &iD_t^\alpha v(x,t)-iu_x+v_{xx}+u-v+\lambda_2 g(|u|^2,|v|^2)v=0, \end{aligned} \tag{1.1}$$

where $0<\alpha\le 1$ is a parameter describing the order of the fractional time. f and g are arbitrary (smooth) nonlinear real functions, and λ_1,λ_2 are parameters. We do not pay attention to the boundary condition in this paper; hence the solution is considered to be either periodic or compactly supported.

☆ This work is supported by the National Natural Science Foundation of China (No. 10971166).

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doi:10.1016/j.camwa.2012.07.004

The time fractional derivative in Eq. (1.1), using the Caputo fractional partial derivative of order α, is defined as [15]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha} & \text{if } 0 < \alpha < 1, \\ \dfrac{\partial u(x,t)}{\partial t} & \text{if } \alpha = 1, \end{cases} \tag{1.2}$$

here $\Gamma(\cdot)$ is the Gamma function.

The discontinuous Galerkin finite element method is a very attractive method for partial differential equations because of its flexibility and efficiency in terms of mesh and shape functions, and a higher order of convergence can be achieved without many iterations. The purpose of the present paper is to develop an implicit fully discrete local discontinuous Galerkin method for these systems. This development is based on the extensive work on DG for problems founded in classic calculus [16,14,17,18]. By choosing the numerical fluxes carefully we prove that our scheme is unconditionally stable.

The rest of this paper is organized as follows. First we introduce some basic notations and mathematical preliminaries, then in Section 3, we discuss the LDG scheme for the fractional equation (1.1), and prove that the scheme is unconditionally stable, and the numerical solution is convergent. Numerical experiments to illustrate the accuracy and capability of the method are given in Section 4. Finally in Section 5 concluding remarks are provided.

2. Notations and auxiliary results

Given a spatial grid $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b$, define the mesh $I_j = \left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right]$, for $j = 1, \ldots N$, and the cell lengths $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $1 \le j \le N$, and $h = \max_{1\le j\le N} \Delta x_j$.

The solution of the numerical scheme is denoted by u_h^n which belongs to the finite element space V_h^k:

$$V_h^k = \{v : v \in P^k(I_j), x \in I_j, j = 1, 2, \ldots, N\},$$

where $P^k(I_j)$ denotes the set of all polynomials of degree at most k in I_j.

For a function $u_h^n \in V_h^k$, we denote the limits at the points $\left\{x_{j+\frac{1}{2}}\right\}$ by

$$(u_h^n)^\pm_{j+\frac{1}{2}} = \lim_{x \to x^\pm_{j+\frac{1}{2}}} u_h^n,$$

and the jump $(u_h^n)^+_{j+\frac{1}{2}} - (u_h^n)^-_{j+\frac{1}{2}}$ by $[u_h^n]_{j+\frac{1}{2}}$. The jump will be zero for a continuous function.

For the error estimates, we shall use the standard projection of a function $\omega(x)$ with $k+1$ continuous derivatives into space V_h^k, denoted by $\mathcal{P}$, i.e., for each j,

$$\int_{I_j} (\mathcal{P}\omega(x) - \omega(x)) v(x) = 0, \quad \forall v \in P^k(I_j), \tag{2.1}$$

and special projection $\mathcal{P}^\pm$ into space V_h^k, i.e., for each j,

$$\begin{aligned} &\int_{I_j} (\mathcal{P}^+\omega(x) - \omega(x)) v(x) = 0, \quad \forall v \in P^{k-1}(I_j), \\ &\mathcal{P}^+\omega\left(x^+_{j-\frac{1}{2}}\right) = \omega\left(x_{j-\frac{1}{2}}\right), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\int_{I_j} (\mathcal{P}^-\omega(x) - \omega(x)) v(x) = 0, \quad \forall v \in P^{k-1}(I_j), \\ &\mathcal{P}^-\omega\left(x^-_{j+\frac{1}{2}}\right) = \omega\left(x_{j+\frac{1}{2}}\right). \end{aligned} \tag{2.3}$$

For the two projections, the following inequality holds [19,20]

$$\|\omega^e\| + h\|\omega^e\|_\infty + h^{\frac{1}{2}}\|\omega^e\|_{\tau_h} \le Ch^{k+1}, \tag{2.4}$$

where $\omega^e = \mathcal{P}\omega - \omega$ or $\omega^e = \mathcal{P}^\pm\omega - \omega$. $\|\omega^e\|_{\tau_h}$ denotes the L^2-norm of ω^e on τ_h, which is

$$\|\omega^e\|_{\tau_h} = \left(\sum_{j=0}^N \left((\omega^e)^+_{j+\frac{1}{2}}\right)^2 + \left((\omega^e)^-_{j+\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}.$$

Here and below we use C to denote a positive constant which may have a different value in each occurrence, and a norm $\|\cdot\|$ denotes the L^2 norm on $\Omega = [a, b]$.

3. The fully discrete LDG scheme

In the following we shall introduce the numerical scheme for system (1.1). Let $\Delta t = T/M$ be the time meshsize, M is a positive integer, $t_n = n\Delta t$, $n = 0, 1, \dots, M$ be mesh points. An approximation to time fractional derivative (1.2) can be obtained by the simple quadrature formula given as [21]

$$D_t^\alpha \omega(x, t_n) = \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} b_i \frac{\omega(x, t_{n-i}) - \omega(x, t_{n-i-1})}{\Delta t} + \gamma^n(\omega), \tag{3.1}$$

where $b_i = (i+1)^{1-\alpha} - i^{1-\alpha}$, $\gamma^n(\omega) \le C(\Delta t)^{2-\alpha}$, C is dependent on ω, T, α.

We know

$$\begin{aligned} &1 = b_0 > b_1 > b_2 > \cdots > b_n > 0, \quad b_n \to 0 (n \to \infty), \\ &\sum_{i=1}^{n} (b_{i-1} - b_i) + b_n = 1. \end{aligned} \tag{3.2}$$

First we decompose the complex functions $u(x, t)$ and $v(x, t)$ into their real and imaginary parts. Setting $u(x, t) = p(x, t) + iq(x, t)$ and $v(x, t) = r(x, t) + is(x, t)$ in system (1.1), we can obtain the following coupled system

$$\begin{aligned} &D_t^\alpha p(x, t) + p_x + q_{xx} + q + s + \lambda_1 f(|u|^2, |v|^2) q(x, t) = 0, \\ &D_t^\alpha q(x, t) + q_x - p_{xx} - (p + r) - \lambda_1 f(|u|^2, |v|^2) p(x, t) = 0, \\ &D_t^\alpha r(x, t) - r_x + s_{xx} + q - s + \lambda_2 g(|u|^2, |v|^2) s(x, t) = 0, \\ &D_t^\alpha s(x, t) - s_x - r_{xx} + r - p - \lambda_2 g(|u|^2, |v|^2) r(x, t) = 0. \end{aligned} \tag{3.3}$$

To define a fully discrete LDG scheme, we rewrite the above system (3.3) as the first-order one

$$\begin{aligned} &\frac{\partial^\alpha p(x, t)}{\partial t^\alpha} + p_x + \rho_x + (q + s) + \lambda_1 f(|u|^2, |v|^2) q(x, t) = 0, \\ &\rho - q_x = 0, \\ &\frac{\partial^\alpha q(x, t)}{\partial t^\alpha} + q_x - z_x - (p + r) - \lambda_1 f(|u|^2, |v|^2) p(x, t) = 0, \\ &z - p_x = 0, \\ &\frac{\partial^\alpha r(x, t)}{\partial t^\alpha} - r_x + \theta_x + (q - s) + \lambda_2 g(|u|^2, |v|^2) s(x, t) = 0, \\ &\theta - s_x = 0, \\ &\frac{\partial^\alpha s(x, t)}{\partial t^\alpha} - s_x - w_x + (r - p) - \lambda_2 g(|u|^2, |v|^2) r(x, t) = 0, \\ &w - r_x = 0. \end{aligned} \tag{3.4}$$

For convenience, we introduce the following notations

$$\begin{aligned} &\mathbf{B}(\omega^n, \omega^n, \mu^n, \phi^n + \psi^n, \lambda_1 f(|u_h^n|^2, |v_h^n|^2)\phi^n; \varphi) \\ &\quad = \int_\Omega \omega^n \varphi dx - \beta \left(\int_\Omega \omega^n \varphi_x dx - \sum_{j=1}^{N} \left((\widehat{\omega^n} \varphi^-)_{j+\frac{1}{2}} - (\widehat{\omega^n} \varphi^+)_{j-\frac{1}{2}} \right) \right) \\ &\qquad - \beta \left(\int_\Omega \mu^n \varphi_x dx - \sum_{j=1}^{N} \left((\widehat{\mu^n} \varphi^-)_{j+\frac{1}{2}} - (\widehat{\mu^n} \varphi^+)_{j-\frac{1}{2}} \right) \right) \\ &\qquad + \beta \int_\Omega (\phi^n + \psi^n) \varphi dx + \beta\lambda_1 \int_\Omega f(|u_h^n|^2, |v_h^n|^2) \phi^n \varphi dx - \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega \omega^{n-i} \varphi dx - b_{n-1} \int_\Omega \omega^0 \varphi dx, \\ &\mathbf{C}(\omega^n, \phi^n; \eta) = \int_\Omega \omega^n \eta dx + \int_\Omega \phi^n w_x dx - \sum_{j=1}^{N} \left((\widehat{\phi^n} \eta^-)_{j+\frac{1}{2}} - (\widehat{\phi^n} \eta^+)_{j-\frac{1}{2}} \right), \end{aligned} \tag{3.5}$$

where $\beta = (\Delta t)^\alpha \Gamma(2 - \alpha)$.

Let $p_h^n, \rho_h^n, q_h^n, z_h^n, r_h^n, \theta_h^n, s_h^n, w_h^n \in V_h^k$ be the approximation of $p(\cdot,t_n), \rho(\cdot,t_n), q(\cdot,t_n), z(\cdot,t_n), r(\cdot,t_n), \theta(\cdot,t_n), s(\cdot,t_n), w(\cdot,t_n)$, respectively. Then an implicit fully discrete LDG scheme can be defined as follows: find $p_h^n, \rho_h^n, q_h^n, z_h^n, r_h^n, \theta_h^n, s_h^n, w_h^n \in V_h^k$, such that for all test functions $\varphi, \psi, \xi, \eta, \delta, \kappa, \phi, \chi \in V_h^k$,

$$\begin{aligned}
&\mathbf{B}(p_h^n, p_h^n, \rho_h^n, q_h^n + s_h^n, \lambda_1 f(|u_h^n|^2, |v_h^n|^2) q_h^n; \varphi) = 0,\\
&\quad \mathbf{C}(\rho_h^n, q_h^n; \psi) = 0,\\
&\mathbf{B}(q_h^n, q_h^n, -z_h^n, -(p_h^n + r_h^n), -\lambda_1 f(|u_h^n|^2, |v_h^n|^2) p_h^n; \xi) = 0,\\
&\quad \mathbf{C}(z_h^n, p_h^n; \eta) = 0,\\
&\mathbf{B}(r_h^n, -r_h^n, \theta_h^n, (q_h^n - s_h^n), \lambda_2 g(|u_h^n|^2, |v_h^n|^2) s_h^n; \delta) = 0,\\
&\quad \mathbf{C}(\theta_h^n, s_h^n; \kappa) = 0,\\
&\mathbf{B}(s_h^n, -s_h^n, -w_h^n, (r_h^n - p_h^n), -\lambda_2 g(|u_h^n|^2, |v_h^n|^2) r_h^n; \phi) = 0,\\
&\quad \mathbf{C}(w_h^n, r_h^n; \chi) = 0.
\end{aligned} \tag{3.6}$$

The "hat" terms in (3.6) in the cell boundary terms from integration by parts are the so-called "numerical fluxes", in order to ensure stability, we can take the following choices simply

$$\begin{aligned}
&\widehat{p_h^n} = (p_h^n)^-, \quad \widehat{q_h^n} = (q_h^n)^-, \quad \widehat{\rho_h^n} = (\rho_h^n)^+, \quad \widehat{z_h^n} = (z_h^n)^+,\\
&\widehat{r_h^n} = (r_h^n)^+, \quad \widehat{s_h^n} = (s_h^n)^+, \quad \widehat{\theta_h^n} = (\theta_h^n)^-, \quad \widehat{w_h^n} = (w_h^n)^-.
\end{aligned} \tag{3.7}$$

We remark that the choice for the fluxes (3.7) is not unique. In fact the crucial part is taking $\widehat{p_h^n}$ and $\widehat{\rho_h^n}$ from opposite sides, $\widehat{q_h^n}$ and $\widehat{z_h^n}$ from opposite sides, $\widehat{r_h^n}$ and $\widehat{\theta_h^n}$ from opposite sides, and $\widehat{s_h^n}$ and $\widehat{w_h^n}$ from opposite sides [22,23].

Since the problem is nonlinear, we would use an iterative method when computing. The definition of the algorithm is now complete.

Now we will consider the stability and convergence of the scheme (3.6), and we have the following result.

Theorem 3.1. *Suppose $u(x,t) = p(x,t) + iq(x,t)$, $v(x,t) = r(x,t) + is(x,t)$ and let $p_h^n, q_h^n, r_h^n, s_h^n \in V_h^k$ be the approximation of $p(\cdot,t_n), q(\cdot,t_n), r(\cdot,t_n), s(\cdot,t_n)$, respectively, then for periodic or compactly supported boundary conditions, the fully-discrete LDG scheme (3.6) is unconditionally stable, and the numerical solutions $p_h^n, q_h^n, r_h^n, s_h^n$ satisfy*

$$\begin{aligned}
&\|p_h^n\|^2 + \|q_h^n\|^2 + \|r_h^n\|^2 + \|s_h^n\|^2 + \beta \sum_{j=1}^{N} ([p_h^n]^2 + [q_h^n]^2 + [r_h^n]^2 + [s_h^n]^2)_{j-\frac{1}{2}}\\
&\quad \le \|p_h^0\|^2 + \|q_h^0\|^2 + \|r_h^0\|^2 + \|s_h^0\|^2, \quad n = 1, 2 \ldots, M.
\end{aligned} \tag{3.8}$$

Proof. Taking the test functions $\varphi = p_h^n, \psi = \beta z_h^n, \xi = q_h^n, \eta = -\beta \rho_h^n, \delta = r_h^n, \kappa = \beta w_h^n, \phi = s_h^n, \chi = -\beta\theta_h^n$ in scheme (3.6), and with the choice of fluxes (3.7) we obtain

$$\begin{aligned}
&\|p_h^n\|^2 + \|q_h^n\|^2 + \|r_h^n\|^2 + \|s_h^n\|^2 + \frac{\beta}{2} \sum_{j=1}^{N} ([p_h^n]^2 + [q_h^n]^2 + [r_h^n]^2 + [s_h^n]^2)_{j-\frac{1}{2}}\\
&\quad + \beta \sum_{j=1}^{N} (\Psi(\rho_h^n, p_h^n, z_h^n, q_h^n, \theta_h^n, r_h^n, w_h^n, s_h^n)_{j+\frac{1}{2}} - \Psi(\rho_h^n, p_h^n, z_h^n, q_h^n, \theta_h^n, r_h^n, w_h^n, s_h^n)_{j-\frac{1}{2}}\\
&\quad + \Theta(\rho_h^n, p_h^n, z_h^n, q_h^n, \theta_h^n, r_h^n, w_h^n, s_h^n)_{j-\frac{1}{2}})\\
&= \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_{\Omega} (p_h^{n-i} p_h^n + q_h^{n-i} q_h^n + r_h^{n-i} r_h^n + s_h^{n-i} s_h^n) dx\\
&\quad + b_{n-1} \int_{\Omega} (p_h^0 p_h^n + q_h^0 q_h^n + r_h^0 r_h^n + s_h^0 s_h^n) dx,
\end{aligned} \tag{3.9}$$

here

$$
\begin{aligned}
\Psi(\rho_h^n, p_h^n, z_h^n, q_h^n, \theta_h^n, r_h^n, w_h^n, s_h^n) &= -(\rho_h^n)^-(p_h^n)^- + \widehat{\rho_h^n}(p_h^n)^- + \widehat{p_h^n}(\rho_h^n)^- \\
&\quad + (q_h^n)^-(z_h^n)^- - \widehat{q_h^n}(z_h^n)^- - \widehat{z_h^n}(q_h^n)^- - (\theta_h^n)^-(r_h^n)^- + \widehat{\theta_h^n}(r_h^n)^- \\
&\quad + \widehat{r_h^n}(\theta_h^n)^- + (w_h^n)^-(s_h^n)^- - \widehat{w_h^n}(s_h^n)^- - \widehat{s_h^n}(w_h^n)^-, \\
\Theta(\rho_h^n, p_h^n, z_h^n, q_h^n, \theta_h^n, r_h^n, w_h^n, s_h^n) &= -(\rho_h^n)^-(p_h^n)^- + (\rho_h^n)^+(p_h^n)^+ + \widehat{\rho_h^n}(p_h^n)^- - \widehat{\rho_h^n}(p_h^n)^+ \\
&\quad + \widehat{p_h^n}(\rho_h^n)^- - \widehat{p_h^n}(\rho_h^n)^+ + (q_h^n)^-(z_h^n)^- - (q_h^n)^+(z_h^n)^+ - \widehat{q_h^n}(z_h^n)^- + \widehat{q_h^n}(z_h^n)^+ \\
&\quad - \widehat{z_h^n}(q_h^n)^- + \widehat{z_h^n}(q_h^n)^+ - (\theta_h^n)^-(r_h^n)^- + (\theta_h^n)^+(r_h^n)^+ + \widehat{\theta_h^n}(r_h^n)^- - \widehat{\theta_h^n}(r_h^n)^+ \\
&\quad + \widehat{r_h^n}(\theta_h^n)^- - \widehat{r_h^n}(\theta_h^n)^+ + (w_h^n)^-(s_h^n)^- - (w_h^n)^+(s_h^n)^+ - \widehat{w_h^n}(s_h^n)^- + \widehat{w_h^n}(s_h^n)^+ \\
&\quad - \widehat{s_h^n}(w_h^n)^- + \widehat{s_h^n}(w_h^n)^+.
\end{aligned}
\tag{3.10}
$$

If we take the fluxes (3.7), and after some manual calculations, we can easily obtain

$$\Theta(\rho_h^n, p_h^n, z_h^n, q_h^n, \theta_h^n, r_h^n, w_h^n, s_h^n) = 0.$$

Then based on Eq. (3.9), we can get

$$
\begin{aligned}
&\|p_h^n\|^2 + \|q_h^n\|^2 + \|r_h^n\|^2 + \|s_h^n\|^2 + \frac{\beta}{2}\sum_{j=1}^{N}([p_h^n]^2 + [q_h^n]^2 + [r_h^n]^2 + [s_h^n]^2)_{j-\frac{1}{2}} \\
&= \sum_{i=1}^{n-1}(b_{i-1} - b_i)\int_\Omega (p_h^{n-i}p_h^n + q_h^{n-i}q_h^n + r_h^{n-i}r_h^n + s_h^{n-i}s_h^n)dx + b_{n-1}\int_\Omega (p_h^0p_h^n + q_h^0q_h^n + r_h^0r_h^n + s_h^0s_h^n)dx.
\end{aligned}
\tag{3.11}
$$

We will prove Theorem 3.1 by mathematical induction. When $n = 1$, from Eq. (3.11) and using Hölder's and Young's inequalities, we can get

$$
\begin{aligned}
&\|p_h^1\|^2 + \|q_h^1\|^2 + \|r_h^1\|^2 + \|s_h^1\|^2 + \frac{\beta}{2}\sum_{j=1}^{N}([p_h^1]^2 + [q_h^1]^2 + [r_h^1]^2 + [s_h^1]^2)_{j-\frac{1}{2}} \\
&\le \frac{1}{2}(\|p_h^0\|^2 + \|q_h^0\|^2 + \|r_h^0\|^2 + \|s_h^0\|^2) + \frac{1}{2}(\|p_h^1\|^2 + \|q_h^1\|^2 + \|r_h^1\|^2 + \|s_h^1\|^2),
\end{aligned}
\tag{3.12}
$$

then we can get the following inequalities immediately

$$
\begin{aligned}
&\|p_h^1\|^2 + \|q_h^1\|^2 + \|r_h^1\|^2 + \|s_h^1\|^2 + \beta\sum_{j=1}^{N}([p_h^1]^2 + [q_h^1]^2 + [r_h^1]^2 + [s_h^1]^2)_{j-\frac{1}{2}} \\
&\le \|p_h^0\|^2 + \|q_h^0\|^2 + \|r_h^0\|^2 + \|s_h^0\|^2,
\end{aligned}
\tag{3.13}
$$

and

$$\|p_h^1\|^2 + \|q_h^1\|^2 + \|r_h^1\|^2 + \|s_h^1\|^2 \le \|p_h^0\|^2 + \|q_h^0\|^2 + \|r_h^0\|^2 + \|s_h^0\|^2. \tag{3.14}$$

Now suppose the following inequality holds

$$\|p_h^m\|^2 + \|q_h^m\|^2 + \|r_h^m\|^2 + \|s_h^m\|^2 \le \|p_h^0\|^2 + \|q_h^0\|^2 + \|r_h^0\|^2 + \|s_h^0\|^2, \quad m = 1, 2, \ldots, K. \tag{3.15}$$

Let $n = K + 1$ in the inequality (3.11), we can obtain

$$
\begin{aligned}
&\|p_h^{K+1}\|^2 + \|q_h^{K+1}\|^2 + \|r_h^{K+1}\|^2 + \|s_h^{K+1}\|^2 + \frac{\beta}{2}\sum_{j=1}^{N}([p_h^{K+1}]^2 + [q_h^{K+1}]^2 + [r_h^{K+1}]^2 + [s_h^{K+1}]^2)_{j-\frac{1}{2}} \\
&\le \sum_{i=1}^{K}(b_{i-1} - b_i)(\|p_h^{K+1-i}\|\|p_h^{K+1}\| + \|q_h^{K+1-i}\|\|q_h^{K+1}\| + \|r_h^{K+1-i}\|\|r_h^{K+1}\| + \|s_h^{K+1-i}\|\|s_h^{K+1}\|) \\
&\quad + b_K(\|p_h^0\|\|p_h^{K+1}\| + \|q_h^0\|\|q_h^{K+1}\| + \|r_h^0\|\|r_h^{K+1}\| + \|s_h^0\|\|s_h^{K+1}\|)
\end{aligned}
$$

$$\leq \frac{1}{2}\sum_{i=1}^{K}(b_{i-1}-b_i)(\|p_h^{K+1-i}\|^2+\|q_h^{K+1-i}\|^2+\|r_h^{K+1-i}\|^2+\|s_h^{K+1-i}\|^2)$$
$$+\frac{1}{2}b_K(\|p_h^0\|^2+\|q_h^0\|^2+\|r_h^0\|^2+\|s_h^0\|^2)+\frac{1}{2}(\|p_h^{K+1}\|^2+\|q_h^{K+1}\|^2+\|r_h^{K+1}\|^2+\|s_h^{K+1}\|^2).$$

Using the assumption (3.15), we can obtain the following inequality easily

$$\|p_h^{K+1}\|^2+\|q_h^{K+1}\|^2+\|r_h^{K+1}\|^2+\|s_h^{K+1}\|^2+\beta\sum_{j=1}^{N}([p_h^{K+1}]^2+[q_h^{K+1}]^2+[r_h^{K+1}]^2+[s_h^{K+1}]^2)_{j-\frac{1}{2}}$$
$$\leq \|p_h^0\|^2+\|q_h^0\|^2+\|r_h^0\|^2+\|s_h^0\|^2.$$

The proof is finished. □

Next we will state the error estimate of the scheme for the linear case, and use (3.7) as our flux choice. We have the following theorem.

Theorem 3.2. *Let $u(x,t_n)$, $v(x,t_n)$ be the exact solution of the linear time-fractional coupled Schrödinger system (1.1), and $p(x,t_n)$, $r(x,t_n)$ and $q(x,t_n)$, $s(x,t_n)$ be their real and imaginary parts, respectively. p_h^n, r_h^n and q_h^n, s_h^n are approximations of $p(x,t_n)$, $r(x,t_n)$ and $q(x,t_n)$, $s(x,t_n)$, respectively, then there exists a positive constant C, such that the following error estimate holds*

$$\|p(x,t_n)-p_h^n\|+\|q(x,t_n)-q_h^n\|+\|r(x,t_n)-r_h^n\|+\|s(x,t_n)-s_h^n\|\leq C\left(h^{k+1}+(\Delta t)^2+(\Delta t)^{\frac{\alpha}{2}}h^{k+\frac{1}{2}}\right). \tag{3.16}$$

Proof. We consider the following linear time-fractional coupled Schrödinger system

$$\begin{aligned}
&i\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+iu_x+u_{xx}+u+v+\lambda_1 u=0,\\
&i\frac{\partial^\alpha v(x,t)}{\partial t^\alpha}-iv_x+v_{xx}+u-v+\lambda_2 v=0.
\end{aligned} \tag{3.17}$$

It is easy to verify that the exact solution of the above PDE (3.17) satisfies

$$\begin{aligned}
&\mathbf{B}(p(x,t_n),p(x,t_n),\rho(x,t_n),q(x,t_n)+s(x,t_n),\lambda_1 q(x,t_n);\varphi)+\beta\int_\Omega \gamma^n(p)\varphi dx=0,\\
&\quad\mathbf{C}(\rho(x,t_n),q(x,t_n);\psi)=0,\\
&\mathbf{B}(q(x,t_n),q(x,t_n),-z(x,t_n),-(p(x,t_n)+r(x,t_n)),-\lambda_1 p(x,t_n);\xi)+\beta\int_\Omega \gamma^n(q)\xi dx=0,\\
&\quad\mathbf{C}(z(x,t_n),p(x,t_n);\eta)=0,\\
&\mathbf{B}(r(x,t_n),-r(x,t_n),\theta(x,t_n),(q(x,t_n)-s(x,t_n)),\lambda_2 s(x,t_n);\delta)+\beta\int_\Omega \gamma^n(r)\delta dx=0,\\
&\quad\mathbf{C}(\theta(x,t_n),s(x,t_n);\kappa)=0,\\
&\mathbf{B}(s(x,t_n),-s(x,t_n),-w(x,t_n),(r(x,t_n)-p(x,t_n)),-\lambda_2 r(x,t_n);\phi)+\beta\int_\Omega \gamma^n(s)\phi dx=0,\\
&\quad\mathbf{C}(w(x,t_n),r(x,t_n);\chi)=0.
\end{aligned} \tag{3.18}$$

We denote

$$\begin{aligned}
e_p^n&=p(x,t_n)-p_h^n=\mathcal{P}^-e_p^n-(\mathcal{P}^-p(x,t_n)-p(x,t_n)),\\
e_\rho^n&=\rho(x,t_n)-\rho_h^n=\mathcal{P}e_\rho^n-(\mathcal{P}\rho(x,t_n)-\rho(x,t_n)),\\
e_q^n&=q(x,t_n)-q_h^n=\mathcal{P}^-e_q^n-(\mathcal{P}^-q(x,t_n)-q(x,t_n)),\\
e_z^n&=z(x,t_n)-z_h^n=\mathcal{P}e_z^n-(\mathcal{P}z(x,t_n)-z(x,t_n)),\\
e_r^n&=r(x,t_n)-r_h^n=\mathcal{P}^+e_r^n-(\mathcal{P}^+r(x,t_n)-r(x,t_n)),\\
e_\theta^n&=\theta(x,t_n)-\theta_h^n=\mathcal{P}e_\theta^n-(\mathcal{P}\theta(x,t_n)-\theta(x,t_n)),\\
e_s^n&=s(x,t_n)-s_h^n=\mathcal{P}^+e_s^n-(\mathcal{P}^+s(x,t_n)-s(x,t_n)),\\
e_w^n&=w(x,t_n)-w_h^n=\mathcal{P}e_w^n-(\mathcal{P}w(x,t_n)-w(x,t_n)).
\end{aligned} \tag{3.19}$$

Subtracting (3.6) from (3.18), we can obtain the error equation

$$
\begin{aligned}
&\mathbf{B}(\mathcal{P}^-e_p^n, \mathcal{P}^-e_p^n, \mathcal{P}e_\rho^n, \mathcal{P}^-e_q^n + \mathcal{P}^+e_s^n, \lambda_1\mathcal{P}^-e_q^n; \varphi) + \mathbf{C}(\mathcal{P}e_\rho^n, \mathcal{P}^-e_q^n; \psi) \\
&\quad + \mathbf{B}(\mathcal{P}^-e_q^n, \mathcal{P}^-e_q^n, -\mathcal{P}e_z^n, -(\mathcal{P}^-e_p^n + \mathcal{P}^+e_r^n), -\lambda_1\mathcal{P}^-e_p^n; \xi) + \mathbf{C}(\mathcal{P}e_z^n, \mathcal{P}^-e_p^n; \eta) \\
&\quad + \mathbf{B}(\mathcal{P}^+e_r^n, -\mathcal{P}^+e_r^n, \mathcal{P}e_\theta^n, (\mathcal{P}^-e_q^n - \mathcal{P}^+e_s^n), \lambda_2\mathcal{P}^+e_s^n; \delta) + \mathbf{C}(\mathcal{P}e_\theta^n, \mathcal{P}^+e_s^n; \kappa) \\
&\quad + \mathbf{B}(\mathcal{P}^+e_s^n, -\mathcal{P}^+e_s^n, -\mathcal{P}e_w^n, (\mathcal{P}^+e_r^n - \mathcal{P}^-e_p^n), -\lambda_2\mathcal{P}^+e_r^n; \phi) + \mathbf{C}(\mathcal{P}e_w^n, \mathcal{P}^+e_r^n; \chi) \\
&= \mathbf{B}(\mathcal{P}^-p(x,t_n) - p(x,t_n), \mathcal{P}^-p(x,t_n) - p(x,t_n), \mathcal{P}\rho(x,t_n) - \rho(x,t_n), \\
&\quad \mathcal{P}^-q(x,t_n) - q(x,t_n) + \mathcal{P}^+s(x,t_n) - s(x,t_n), \lambda_1(\mathcal{P}^-q(x,t_n) - q(x,t_n)); \varphi) \\
&\quad + \mathbf{C}(\mathcal{P}\rho(x,t_n) - \rho(x,t_n), \mathcal{P}^-q(x,t_n) - q(x,t_n); \psi) \\
&\quad + \mathbf{B}(\mathcal{P}^-q(x,t_n) - q(x,t_n), \mathcal{P}^-q(x,t_n) - q(x,t_n), -(\mathcal{P}z(x,t_n) - z(x,t_n)), \\
&\quad - (\mathcal{P}^-p(x,t_n) - p(x,t_n) + \mathcal{P}^+r(x,t_n) - r(x,t_n)), -\lambda_1(\mathcal{P}^-p(x,t_n) - p(x,t_n)); \xi) \\
&\quad + \mathbf{C}(\mathcal{P}z(x,t_n) - z(x,t_n), \mathcal{P}^-p(x,t_n) - p(x,t_n); \eta) \\
&\quad + \mathbf{B}(\mathcal{P}^+r(x,t_n) - r(x,t_n), -(\mathcal{P}^+r(x,t_n) - r(x,t_n)), \mathcal{P}\theta(x,t_n) - \theta(x,t_n), \\
&\quad (\mathcal{P}^-q(x,t_n) - q(x,t_n) - (\mathcal{P}^+s(x,t_n) - s(x,t_n))), \lambda_2(\mathcal{P}^+s(x,t_n) - s(x,t_n)); \delta) \\
&\quad + \mathbf{C}(\mathcal{P}\theta(x,t_n) - \theta(x,t_n), \mathcal{P}^+s(x,t_n) - s(x,t_n); \kappa) \\
&\quad + \mathbf{B}(\mathcal{P}^+s(x,t_n) - s(x,t_n), -(\mathcal{P}^+s(x,t_n) - s(x,t_n)), -(\mathcal{P}w(x,t_n) - w(x,t_n)), \\
&\quad (\mathcal{P}^+r(x,t_n) - r(x,t_n) - (\mathcal{P}^-p(x,t_n) - p(x,t_n))), -\lambda_2(\mathcal{P}^+r(x,t_n) - r(x,t_n)); \phi) \\
&\quad + \mathbf{C}(\mathcal{P}w(x,t_n) - w(x,t_n), \mathcal{P}^+r(x,t_n) - r(x,t_n); \chi) \\
&\quad - \beta\left(\int_\Omega \gamma^n(p)\varphi dx + \int_\Omega \gamma^n(q)\xi dx + \int_\Omega \gamma^n(r)\delta dx + \int_\Omega \gamma^n(s)\phi dx\right).
\end{aligned}
\tag{3.20}
$$

Denoting the left-hand and right-hand terms in (3.20) by LHT and RHT, respectively. With the fluxes (3.7), and taking the test functions $\varphi = \mathcal{P}^-e_p^n, \psi = \beta\mathcal{P}e_z^n, \xi = \mathcal{P}^-e_q^n, \eta = -\beta\mathcal{P}e_\rho^n, \delta = \mathcal{P}^+e_r^n, \kappa = \beta\mathcal{P}e_w^n, \phi = \mathcal{P}^+e_s^n, \chi = -\beta\mathcal{P}e_\theta^n$ in (3.20). First we consider the left-hand term LHT in (3.20), almost the same as that used for the equality (3.11), and we have

$$
\begin{aligned}
LHT &= \|\mathcal{P}^-e_p^n\|^2 + \|\mathcal{P}^-e_q^n\|^2 + \|\mathcal{P}^+e_r^n\|^2 + \|\mathcal{P}^+e_s^n\|^2 + \frac{\beta}{2}\sum_{j=1}^N([\mathcal{P}^-e_p^n]^2 + [\mathcal{P}^-e_q^n]^2 + [\mathcal{P}^+e_r^n]^2 + [\mathcal{P}^+e_s^n]^2)_{j-\frac{1}{2}} \\
&\quad - \sum_{i=1}^{n-1}(b_{i-1} - b_i)\int_\Omega (\mathcal{P}^-e_p^{n-i}\mathcal{P}^-e_p^n + \mathcal{P}^-e_q^{n-i}\mathcal{P}^-e_q^n + \mathcal{P}^+e_r^{n-i}\mathcal{P}^+e_r^n + \mathcal{P}^+e_s^{n-i}\mathcal{P}^+e_s^n)dx \\
&\quad - b_{n-1}\int_\Omega (\mathcal{P}^-e_p^0\mathcal{P}^-e_p^n + \mathcal{P}^-e_q^0\mathcal{P}^-e_q^n + \mathcal{P}^+e_r^0\mathcal{P}^+e_r^n + \mathcal{P}^+e_s^0\mathcal{P}^+e_s^n)dx.
\end{aligned}
\tag{3.21}
$$

Now we consider the right-hand term RHT in (3.20). Using the properties (2.1) and (2.3), we can find that some terms are equal to zeros immediately, we have

$$
\begin{aligned}
RHT &= \int_\Omega (\mathcal{P}^-p(x,t_n) - p(x,t_n))\mathcal{P}^-e_p^n dx + \int_\Omega (\mathcal{P}^-q(x,t_n) - q(x,t_n))\mathcal{P}^-e_q^n dx \\
&\quad + \int_\Omega (\mathcal{P}^+s(x,t_n) - s(x,t_n))\mathcal{P}^+e_s^n dx + \int_\Omega (\mathcal{P}^+r(x,t_n) - r(x,t_n))\mathcal{P}^+e_r^n dx \\
&\quad - \beta\sum_{j=1}^N(\mathcal{P}\rho(x,t_n) - \rho(x,t_n))^+[\mathcal{P}^-e_p^n]_{j-\frac{1}{2}} + \beta\sum_{j=1}^N(\mathcal{P}z(x,t_n) - z(x,t_n))^+[\mathcal{P}^-e_q^n]_{j-\frac{1}{2}} \\
&\quad - \beta\sum_{j=1}^N(\mathcal{P}\theta(x,t_n) - \theta(x,t_n))^+[\mathcal{P}^+e_r^n]_{j-\frac{1}{2}} + \beta\sum_{j=1}^N(\mathcal{P}w(x,t_n) - w(x,t_n))^+[\mathcal{P}^+e_s^n]_{j-\frac{1}{2}} \\
&\quad + \beta\int_\Omega (\mathcal{P}^-q(x,t_n) - q(x,t_n) + \mathcal{P}^+s(x,t_n) - s(x,t_n))\mathcal{P}^-e_p^n dx
\end{aligned}
$$

$$
\begin{aligned}
&-\beta \int_\Omega (\mathcal{P}^- p(x,t_n) - p(x,t_n) - (\mathcal{P}^+ r(x,t_n) - r(x,t_n)))\mathcal{P}^- e_q^n dx \\
&+\beta \int_\Omega (\mathcal{P}^- q(x,t_n) - q(x,t_n) - (\mathcal{P}^+ s(x,t_n) - s(x,t_n)))\mathcal{P}^+ e_r^n dx \\
&+\beta \int_\Omega (\mathcal{P}^+ r(x,t_n) - r(x,t_n) - (\mathcal{P}^- p(x,t_n) - p(x,t_n)))\mathcal{P}^+ e_s^n dx \\
&+\lambda_1 \int_\Omega (\mathcal{P}^- q(x,t_n) - q(x,t_n))\mathcal{P}^- e_p^n dx - \lambda_1 \int_\Omega (\mathcal{P}^- p(x,t_n) - p(x,t_n))\mathcal{P}^- e_q^n dx \\
&+\lambda_2 \int_\Omega (\mathcal{P}^+ s(x,t_n) - s(x,t_n))\mathcal{P}^+ e_r^n dx - \lambda_2 \int_\Omega (\mathcal{P}^+ r(x,t_n) - r(x,t_n))\mathcal{P}^+ e_s^n dx \\
&-\sum_{i=1}^{n-1}(b_{i-1}-b_i)\int_\Omega ((\mathcal{P}^- p(x,t_{n-i}) - p(x,t_{n-i}))\mathcal{P}^- e_p^n + (\mathcal{P}^- q(x,t_{n-i}) - q(x,t_{n-i}))\mathcal{P}^- e_q^n \\
&+(\mathcal{P}^+ r(x,t_{n-i}) - r(x,t_{n-i}))\mathcal{P}^+ e_r^n + (\mathcal{P}^+ s(x,t_{n-i}) - s(x,t_{n-i}))\mathcal{P}^+ e_s^n)dx \\
&-b_{n-1}\int_\Omega ((\mathcal{P}^- p(x,t_0) - p(x,t_0))\mathcal{P}^- e_p^n + (\mathcal{P}^- q(x,t_0) - q(x,t_0))\mathcal{P}^- e_q^n \\
&+(\mathcal{P}^+ r(x,t_0) - r(x,t_0))\mathcal{P}^+ e_r^n + (\mathcal{P}^+ s(x,t_0) - s(x,t_0))\mathcal{P}^+ e_s^n)dx \\
&-\beta\left(\int_\Omega \gamma^n(p)\mathcal{P}^- e_p^n dx + \int_\Omega \gamma^n(q)\mathcal{P}^- e_q^n dx + \int_\Omega \gamma^n(r)\mathcal{P}^+ e_r^n dx + \int_\Omega \gamma^n(s)\mathcal{P}^+ e_s^n dx\right).
\end{aligned}
$$

By using Hölder's and Young's inequalities, and the property (2.4), we know, there exists a positive constant C, such that

$$
\begin{aligned}
RHT \le\ & C\left(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\right)^2 + \frac{1}{4}(\|\mathcal{P}^- e_p^n\|^2 + \|\mathcal{P}^- e_q^n\|^2 + \|\mathcal{P}^+ e_r^n\|^2 + \|\mathcal{P}^+ e_s^n\|^2) \\
&+ \frac{\beta}{4}\sum_{j=1}^N ([\mathcal{P}^- e_p^n]^2 + [\mathcal{P}^- e_q^n]^2 + [\mathcal{P}^+ e_r^n]^2 + [\mathcal{P}^+ e_s^n]^2)_{j-\frac{1}{2}}. \qquad (3.22)
\end{aligned}
$$

Based on the inequality (3.21) and (3.22), we have

$$
\begin{aligned}
&\|\mathcal{P}^- e_p^n\|^2 + \|\mathcal{P}^- e_q^n\|^2 + \|\mathcal{P}^+ e_r^n\|^2 + \|\mathcal{P}^+ e_s^n\|^2 + \frac{\beta}{2}\sum_{j=1}^N ([\mathcal{P}^- e_p^n]^2 + [\mathcal{P}^- e_q^n]^2 + [\mathcal{P}^+ e_r^n]^2 + [\mathcal{P}^+ e_s^n]^2)_{j-\frac{1}{2}} \\
&\le \sum_{i=1}^{n-1}(b_{i-1}-b_i)\int_\Omega (\mathcal{P}^- e_p^{n-i}\mathcal{P}^- e_p^n + \mathcal{P}^- e_q^{n-i}\mathcal{P}^- e_q^n + \mathcal{P}^+ e_r^{n-i}\mathcal{P}^+ e_r^n + \mathcal{P}^+ e_s^{n-i}\mathcal{P}^+ e_s^n)dx \\
&\quad + b_{n-1}\int_\Omega (\mathcal{P}^- e_p^0\mathcal{P}^- e_p^n + \mathcal{P}^- e_q^0\mathcal{P}^- e_q^n + \mathcal{P}^+ e_r^0\mathcal{P}^+ e_r^n + \mathcal{P}^+ e_s^0\mathcal{P}^+ e_s^n)dx + C\left(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\right)^2 \\
&\quad + \frac{1}{4}(\|\mathcal{P}^- e_p^n\|^2 + \|\mathcal{P}^- e_q^n\|^2 + \|\mathcal{P}^+ e_r^n\|^2 + \|\mathcal{P}^+ e_s^n\|^2) \\
&\quad + \frac{\beta}{4}\sum_{j=1}^N ([\mathcal{P}^- e_p^n]^2 + [\mathcal{P}^- e_q^n]^2 + [\mathcal{P}^+ e_r^n]^2 + [\mathcal{P}^+ e_s^n]^2)_{j-\frac{1}{2}}. \qquad (3.23)
\end{aligned}
$$

We prove the error estimate by mathematical induction. When $n = 1$, the inequality (3.23) becomes

$$
\begin{aligned}
&\|\mathcal{P}^- e_p^1\|^2 + \|\mathcal{P}^- e_q^1\|^2 + \|\mathcal{P}^+ e_r^1\|^2 + \|\mathcal{P}^+ e_s^1\|^2 + \frac{\beta}{2}\sum_{j=1}^N ([\mathcal{P}^- e_p^1]^2 + [\mathcal{P}^- e_q^1]^2 + [\mathcal{P}^+ e_r^1]^2 + [\mathcal{P}^+ e_s^1]^2)_{j-\frac{1}{2}} \\
&\le \int_\Omega (\mathcal{P}^- e_p^0\mathcal{P}^- e_p^1 + \mathcal{P}^- e_q^0\mathcal{P}^- e_q^1 + \mathcal{P}^+ e_r^0\mathcal{P}^+ e_r^1 + \mathcal{P}^+ e_s^0\mathcal{P}^+ e_s^1)dx + C\left(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\right)^2 \\
&\quad + \frac{1}{4}(\|\mathcal{P}^- e_p^1\|^2 + \|\mathcal{P}^- e_q^1\|^2 + \|\mathcal{P}^+ e_r^1\|^2 + \|\mathcal{P}^+ e_s^1\|^2) \\
&\quad + \frac{\beta}{4}\sum_{j=1}^N ([\mathcal{P}^- e_p^1]^2 + [\mathcal{P}^- e_q^1]^2 + [\mathcal{P}^+ e_r^1]^2 + [\mathcal{P}^+ e_s^1]^2)_{j-\frac{1}{2}}. \qquad (3.24)
\end{aligned}
$$

It is easy to see that $\|\mathcal{P}^- e^0_\omega\| \le Ch^{k+1}$, $\omega = p, q, r, s$, we can obtain

$$\|\mathcal{P}^- e^1_p\|^2 + \|\mathcal{P}^- e^1_q\|^2 + \|\mathcal{P}^+ e^1_r\|^2 + \|\mathcal{P}^+ e^1_s\|^2 \le C\left(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\right). \tag{3.25}$$

Next we suppose the following inequality holds

$$\|\mathcal{P}^- e^m_p\|^2 + \|\mathcal{P}^- e^m_q\|^2 + \|\mathcal{P}^+ e^m_r\|^2 + \|\mathcal{P}^+ e^m_s\|^2 \le C\left(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\right),$$
$$m = 1, 2, \ldots, K. \tag{3.26}$$

When $n = K + 1$, from Eq. (3.23), we deduce

$$\begin{aligned}
&\|\mathcal{P}^- e^{K+1}_p\|^2 + \|\mathcal{P}^- e^{K+1}_q\|^2 + \|\mathcal{P}^+ e^{K+1}_r\|^2 + \|\mathcal{P}^+ e^{K+1}_s\|^2 \\
&\quad + \frac{\beta}{2}\sum_{j=1}^{N}([\mathcal{P}^- e^{K+1}_p]^2 + [\mathcal{P}^- e^{K+1}_q]^2 + [\mathcal{P}^+ e^{K+1}_r]^2 + [\mathcal{P}^+ e^{K+1}_s]^2)_{j-\frac{1}{2}} \\
&\le \sum_{i=1}^{K}(b_{i-1} - b_i)\int_\Omega (\mathcal{P}^- e^{K+1-i}_p \mathcal{P}^- e^{K+1}_p + \mathcal{P}^- e^{K+1-i}_q \mathcal{P}^- e^{K+1}_q + \mathcal{P}^+ e^{K+1-i}_r \mathcal{P}^+ e^{K+1}_r + \mathcal{P}^+ e^{K+1-i}_s \mathcal{P}^+ e^{K+1}_s)dx \\
&\quad + b_K \int_\Omega (\mathcal{P}^- e^0_p \mathcal{P}^- e^{K+1}_p + \mathcal{P}^- e^0_q \mathcal{P}^- e^{K+1}_q + \mathcal{P}^+ e^0_r \mathcal{P}^+ e^{K+1}_r + \mathcal{P}^+ e^0_s \mathcal{P}^+ e^{K+1}_s)dx \\
&\quad + C\left(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\right)^2 + \frac{1}{4}(\|\mathcal{P}^- e^{K+1}_p\|^2 + \|\mathcal{P}^- e^{K+1}_q\|^2 + \|\mathcal{P}^+ e^{K+1}_r\|^2 + \|\mathcal{P}^+ e^{K+1}_s\|^2) \\
&\quad + \frac{\beta}{4}\sum_{j=1}^{N}([\mathcal{P}^- e^{K+1}_p]^2 + [\mathcal{P}^- e^{K+1}_q]^2 + [\mathcal{P}^+ e^{K+1}_r]^2 + [\mathcal{P}^+ e^{K+1}_s]^2)_{j-\frac{1}{2}} \\
&\le \left(\sum_{i=1}^{K}(b_{i-1} - b_i)\|\mathcal{P}^- e^{K+1-i}_p\| + b_K\|\mathcal{P}^- e^0_p\|\right)^2 + \left(\sum_{i=1}^{K}(b_{i-1} - b_i)\|\mathcal{P}^- e^{K+1-i}_q\| + b_K\|\mathcal{P}^- e^0_q\|\right)^2 \\
&\quad + \left(\sum_{i=1}^{K}(b_{i-1} - b_i)\|\mathcal{P}^+ e^{K+1-i}_r\| + b_K\|\mathcal{P}^+ e^0_r\|\right)^2 + \left(\sum_{i=1}^{K}(b_{i-1} - b_i)\|\mathcal{P}^+ e^{K+1-i}_s\| + b_K\|\mathcal{P}^+ e^0_s\|\right)^2 \\
&\quad + C\left(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\right)^2 + \frac{1}{2}(\|\mathcal{P}^- e^{K+1}_p\|^2 + \|\mathcal{P}^- e^{K+1}_q\|^2 + \|\mathcal{P}^+ e^{K+1}_r\|^2 + \|\mathcal{P}^+ e^{K+1}_s\|^2) \\
&\quad + \frac{\beta}{4}\sum_{j=1}^{N}([\mathcal{P}^- e^{K+1}_p]^2 + [\mathcal{P}^- e^{K+1}_q]^2 + [\mathcal{P}^+ e^{K+1}_r]^2 + [\mathcal{P}^+ e^{K+1}_s]^2)_{j-\frac{1}{2}}.
\end{aligned}$$

Noticing the property (3.2), and using the assumption (3.26), we can get the following result immediately

$$\|\mathcal{P}^- e^{K+1}_p\| + \|\mathcal{P}^- e^{K+1}_q\| + \|\mathcal{P}^+ e^{K+1}_r\| + \|\mathcal{P}^+ e^{K+1}_s\| \le C\left(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\right).$$

Thus Theorem 3.2 follows by the triangle inequality and the interpolating property (2.4). □

4. Numerical examples

In this section we offer some numerical examples to illustrate the accuracy and capability of the method. For this purpose, we calculate the numerical results of the exact solutions (for the cases where exact solutions are available). We mainly focus on the spatial accuracy, so a small time step is used such that errors stemming from the temporal approximation are negligible. With the aid of successive mesh refinements we have verified that the results shown are numerically convergent.

Example 4.1. In this example we show an accuracy test for the nonhomogeneous linear time-fractional coupled Schrödinger system

$$\begin{aligned}
&iD^\alpha_t u(x,t) + iu_x + u_{xx} + u + v + \lambda_1 u = f_1(x,t), \\
&iD^\alpha_t v(x,t) - iv_x + v_{xx} + u - v + \lambda_2 v = f_2(x,t),
\end{aligned} \tag{4.1}$$

and we take the exact solution $u(x,t) = t^2(\cos(2\pi x) + i\sin(2\pi x))$, $v(x,t) = t^2(\cos(2\pi x) + i\sin(2\pi x))$. Choose $\lambda_1 = 1.0$, $\lambda_2 = -2.0$ in the numerical experiment, and a fixed time step $\Delta t = 1/1000$. The solution is computed with a periodic boundary condition in $[0, 1]$ using P^2 elements. In Figs. 1–4, we show that the errors in the L^2-norm and the L^∞-norm attain the third order of accuracy for piecewise P^2 polynomials for two values of α: 0.2 and 0.6. The numerical results are consistent with our theoretical results in Theorem 3.2.

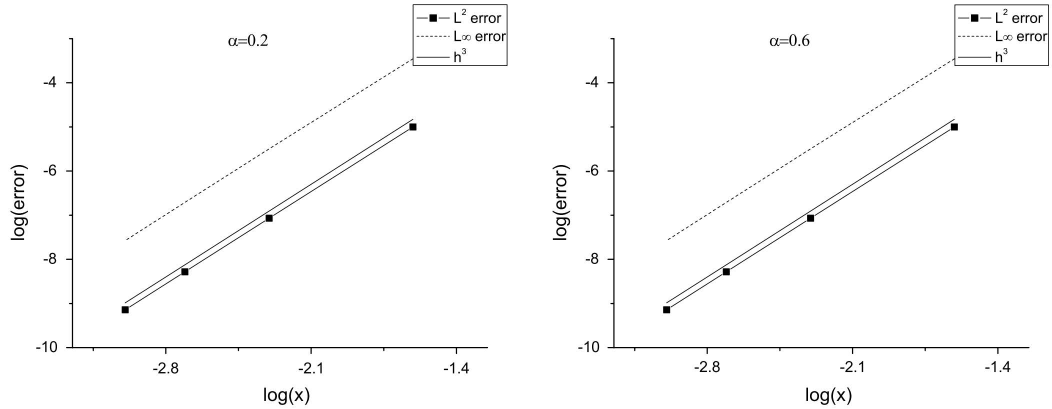

Fig. 1. The convergence rate for the real part p of u when using piecewise P^2 polynomials.

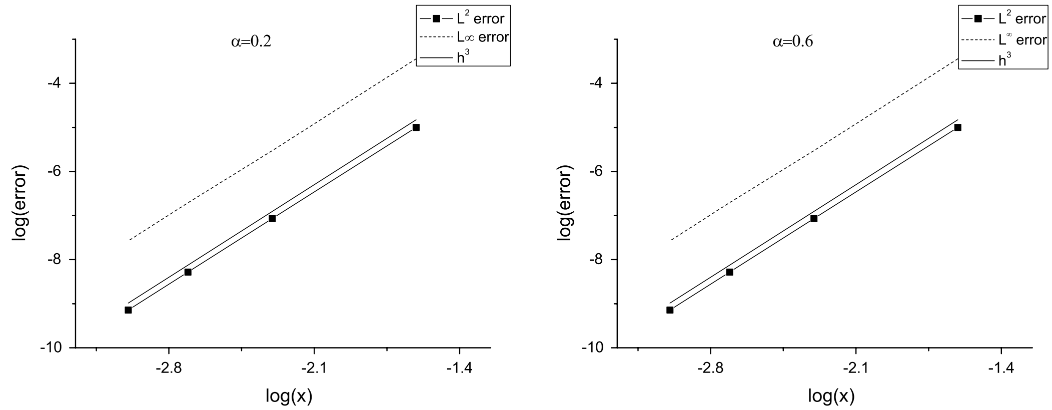

Fig. 2. The convergence rate for the imaginary part q of u when using piecewise P^2 polynomials.

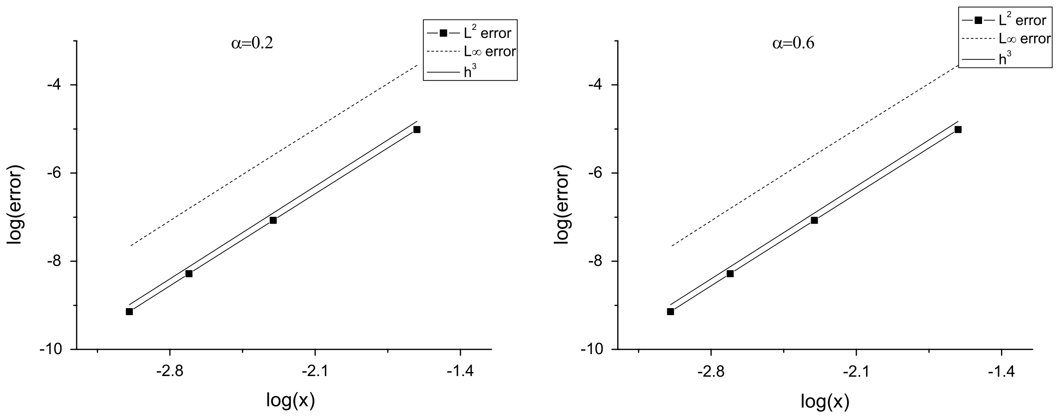

Fig. 3. The convergence rate for the imaginary part r of v when using piecewise P^2 polynomials.

Example 4.2. Consider the following nonhomogeneous time-fractional coupled Schrödinger system in $\Omega = [0, 2\pi]$,

$$\begin{aligned} &iD_t^{\alpha} u(x,t) + iu_x + u_{xx} + u + v + 2(|u|^2 + |v|^2)u = f_1(x,t), \\ &iD_t^{\alpha} v(x,t) - iv_x + v_{xx} + u - v + 4(|u|^2 + |v|^2)v = f_2(x,t), \end{aligned} \tag{4.2}$$

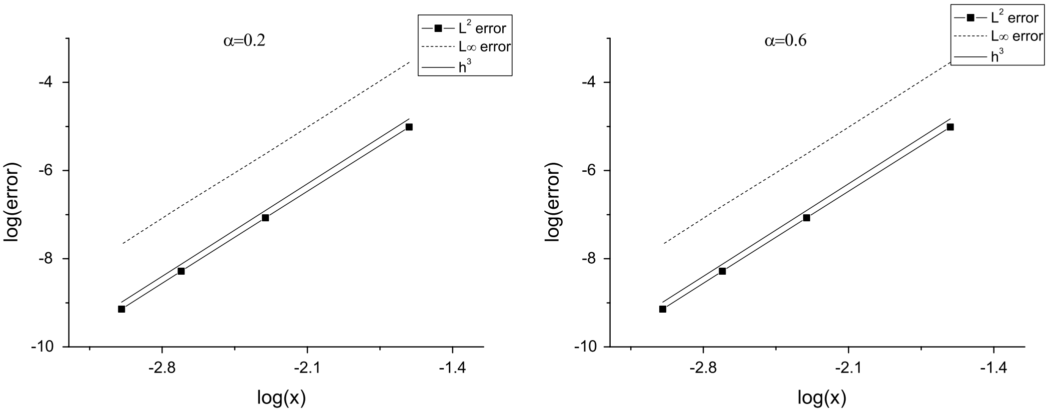

Fig. 4. The convergence rate for the imaginary part s of v when using piecewise P^2 polynomials.

Table 1
The error and the order of convergence of the scheme (3.6) for u using piecewise P^2 elements in Example 4.2, $\Delta t = 1/1000$.

	N	L^2-error	order	L^∞-error	order
Accuracy test for the real part p					
$\alpha = 0.3$	5	2.055338427712268E−002	–	3.940127277638766E−002	–
	10	2.207149081911062E−003	3.22	4.283299134504037E−003	3.20
	15	6.441575279201171E−004	3.04	1.258432300355619E−003	3.02
	20	2.722519904142411E−004	2.99	5.361111602396171E−004	2.97
$\alpha = 0.5$	5	2.025078107603650E−002	–	3.881906918683925E−002	–
	10	2.206526628382518E−003	3.20	4.266660598903182E−003	3.19
	15	6.432488986736275E−004	3.04	1.237338260081475E−003	3.05
	20	2.702576760862203E−004	3.01	5.136282154533662E−004	3.06
$\alpha = 0.7$	5	1.987072553365940E−002	–	3.802649657867085E−002	–
	10	2.210861111363402E−003	3.17	4.218999012138069E−003	3.17
	15	6.600441930718508E−004	2.98	1.180026465226247E−003	3.14
	20	3.084501078083465E−004	2.64	4.623681570881111E−004	3.26
Accuracy test for the imaginary part q					
$\alpha = 0.3$	5	2.055336817647349E−002	–	4.010800141725612E−002	–
	10	2.207147971678494E−003	3.22	4.155672380349367E−003	3.27
	15	6.441564710777718E−004	3.04	1.259242918733167E−003	2.94
	20	2.722499557539329E−004	2.99	5.360885764440202E−004	2.97
$\alpha = 0.5$	5	2.025076461649922E−002	–	3.982325539055442E−002	–
	10	2.206525904688499E−003	3.20	4.131635432924009E−003	3.27
	15	6.432493582948244E−004	3.04	1.239860406883203E−003	2.97
	20	2.702592813612880E−004	3.01	5.135981768213962E−004	3.06
$\alpha = 0.7$	5	1.987070967058925E−002	–	3.935378667275358E−002	–
	10	2.210862226256330E−003	3.17	4.064654578326317E−003	3.28
	15	6.600508221638474E−004	2.98	1.184973452666938E−003	3.04
	20	3.084647387403222E−004	2.64	4.623512057211898E−004	3.27

and the corresponding forcing term $f_1(x,t), f_2(x,t)$ is of the form

$$
\begin{aligned}
f_1(x,t) &= -\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\sin x + 4t^6\cos x + i\left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\cos x + 4t^6\sin x\right),\\
f_2(x,t) &= -\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\sin x + 8t^6\cos x + i\left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\cos x + 8t^6\sin x\right),
\end{aligned}
\tag{4.3}
$$

then the exact solution is $u(x,t) = t^2(\cos x + i\sin x)$, $v(x,t) = t^2(\cos x + i\sin x)$. We take piecewise P^2 polynomials as basis functions, and fix the time step $\Delta t = 1/1000$. The L^2 and L^∞ errors and the numerical orders of accuracy at time $T = 1$ for different α are contained in Tables 1 and 2 for u and v, respectively. We can see that the numerical results are consistent with our theoretical results.

Table 2
The error and the order of convergence of the scheme (3.6) for v using piecewise P^2 elements in Example 4.2, $\Delta t = 1/1000$.

	N	L^2-error	order	L^∞-error	order
Accuracy test for the real part r					
$\alpha = 0.3$	5	1.989761432028803E−002	–	3.363372750643362E−002	–
	10	2.205169862688210E−003	3.17	3.842837176445735E−003	3.13
	15	6.447169582911108E−004	3.03	1.104805315003704E−003	3.07
	20	2.738559629446599E−004	2.98	4.456803945323312E−004	3.16
$\alpha = 0.5$	5	1.978124326931284E−002	–	3.301505805776400E−002	–
	10	2.204788478052998E−003	3.17	3.842079895435002E−003	3.10
	15	6.444823854918444E−004	3.03	1.112918865607646E−003	3.06
	20	2.733792869003352E−004	2.98	4.567687610110911E−004	3.10
$\alpha = 0.7$	5	1.939967760450215E−002	–	3.199757449919333E−002	–
	10	2.208471306238916E−003	3.13	3.901578832120611E−003	3.04
	15	6.581492732998255E−004	2.99	1.185510859952082E−003	2.94
	20	3.045438923477765E−004	2.68	5.331318381103789E−004	2.78
Accuracy test for the imaginary part s					
$\alpha = 0.3$	5	1.989762470988687E−002	–	3.239110838342651E−002	–
	10	2.205169103760043E−003	3.17	3.748911297168472E−003	3.11
	15	6.447169024362816E−004	3.03	1.106439791657921E−003	3.01
	20	2.738563031080162E−004	2.98	4.457702777746226E−004	3.16
$\alpha = 0.5$	5	1.978125959821298E−002	–	3.156640891062024E−002	–
	10	2.204788525014052E−003	3.17	3.759396010132543E−003	3.07
	15	6.444849069713337E−004	3.03	1.111506521325273E−003	3.01
	20	2.733856840875078E−004	2.98	4.568050513618070E−004	3.09
$\alpha = 0.7$	5	1.939969719379127E−002	–	3.045588687378864E−002	–
	10	2.208469608248439E−003	3.13	3.834677253926285E−003	2.99
	15	6.581454787218685E−004	2.99	1.179635311725796E−003	2.91
	20	3.045360283900895E−004	2.68	5.331243511061345E−004	2.76

5. Conclusion

In this paper an implicit fully discrete local discontinuous Galerkin (LDG) finite element method is presented for solving a class of time-fractional coupled Schrödinger systems. The stability is ensured by a careful choice of interface numerical fluxes. We prove that our scheme is unconditionally stable and the L^2 error estimate for the linear case has the convergence rate $O\left(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\right)$. To date we are not aware of any similar results in published papers. Although not addressed in this paper, the scheme can be extended to solve the two or higher dimensional case easily, and the theoretical results are also valid. The results show that the LDG method is a powerful and efficient technique in solving this class of coupled systems with fractional order.

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